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# Vertex operator realization of symplectic and orthogonal $S$-functions

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**Abstract.** It is shown that symplectic and orthogonal Schur functions can be realized by the action of the modes of certain vertex operators on the function **1**. The properties of these vertex operators allow one to reproduce the basic propreties of these types of symmetric functions. The vertex operator modes are shown to obey free fermionic relations which provide applications such as calculating products and plethysms as well as generating Hirota-type partial differential equations (PDEs) which have symplectic $S$-functions as $\tau$ functions.

## 1. Introduction

In this article, we endeavour to construct a vertex operator (VO) realization of symplectic and orthogonal $S$-functions. This has been motivated by two things: analogous vertex operator realizations of other types of symmetric functions, namely normal $S$-functions [1], $Q$-functions [2, 3], Hall–Littlewood functions [4] and Macdonald functions [5–8] and their application to investigating properties of these functions; and the application of vertex operators in the realizations and representations of (quantum) affine algebras [9–16]. For $S$-functions (respectively $Q$-functions), it is the fact that their VO realization is intimately connected (via the boson–fermion correspondence [17]) to the algebra of free fermions (respectively neutral free fermions) that provides a conduit to applications such as explaining the generation of certain hierarchies of nonlinear PDEs (see [1, 18] and references therein), and the derivation of determinantal identities for $S$- and $Q$-functions [3, 19]. For generic Macdonald functions and their various specializations (including Hall–Littlewood functions), the modes of their relevant vertex operators do not enjoy such pleasant properties and hence the above-mentioned applications are harder (if not impossible) to pursue. However, when it comes to symplectic and orthogonal $S$-functions, we shall see that the appropriate vertex operators do indeed yield a simple algebraic structure (like $S$-functions) and hence can be exploited in various applications. This has indeed been previously postulated in a recent article [20].

We begin by establishing notation and reviewing the definitions and basic properties of symplectic and orthogonal Schur functions, including their main determinantal representations, and representations in terms of raising and lowering operators. We then briefly summarize the standard boson–fermion correspondence, emphasizing the fact that ordinary Schur functions can be generated in two equivalent (but distinct) ways, via free fermionic expectation values using Wick's theorem, and also via the action of vertex operator modes on the 'bosonic' vacuum **1**. Turning our attention back to symplectic and orthogonal $S$-functions, we show how these can be generated by the action of certain modified vertex

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operators acting on **1**, and discover that the modes of these vertex operators *also* obey the anti-commutation relations of free fermions. This allows these types of functions to be realized as fermionic expectation values in two different ways: one involving a modified Hamiltonian, and one involving a modified dual vacuum. It is this latter perspective that allows one to generate a hierarchy of Hirota equations which have symplectic $S$-functions as $\tau$ functions. We also include applications to calculating products and (outer) plethysms of symplectic (or orthogonal) $S$-functions.

### *1.1. Symplectic and orthogonal S-functions*

There are several equivalent definitions of symplectic and orthogonal $S$-functions. We shall define them here as the skew of ordinary $S$-functions $s_\lambda$ by certain series of $S$-functions (see [21] for an excellent treatment of these functions and their application to the representation theory of simple Lie algebras). The series $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are defined by

$$\mathcal{A}(x) = \prod_{i<j}(1 - x_i x_j) = \sum_{\alpha\in A}(-1)^{|\alpha|/2} s_\alpha(x) \tag{1.1}$$

$$\mathcal{B}(x) = \prod_{i<j}(1 - x_i x_j)^{-1} = \sum_{\beta\in B} s_\beta(x) \tag{1.2}$$

$$\mathcal{C}(x) = \prod_{i\leqslant j}(1 - x_i x_j) = \sum_{\gamma\in C}(-1)^{|\gamma|/2} s_\gamma(x) \tag{1.3}$$

$$\mathcal{D}(x) = \prod_{i\leqslant j}(1 - x_i x_j)^{-1} = \sum_{\delta\in D} s_\delta(x) \tag{1.4}$$

where $A$ and $C$ are sets of partitions of the form $(a_1, a_2, \ldots | a_1 + 1, a_2 + 1, \ldots)$ and $(a_1 + 1, a_2 + 1, \ldots | a_1, a_2, \ldots)$, respectively, in Frobenius notation, $B$ is the set of partitions whose distinct parts are repeated an even number of times, and $D$ is the set of partitions whose parts are even. We restrict ourselves to the case of a (countably) infinite number of indeterminates $(x_1, x_2, x_3, \ldots)$, in order to avoid having to deal with modification rules [22]. The symplectic and orthogonal $S$-functions $sp_\lambda$, $o_\lambda$ can be defined as

$$\begin{aligned} sp_\lambda &= \langle\lambda\rangle = s_{\lambda/\mathcal{A}} = \sum_{\alpha\in A}(-1)^{|\alpha|/2} s_{\lambda/\alpha} \\ o_\lambda &= [\lambda] = s_{\lambda/\mathcal{C}} = \sum_{\gamma\in C}(-1)^{|\gamma|/2} s_{\lambda/\gamma}. \end{aligned} \tag{1.5}$$

These can be inverted, so that $s_\lambda = sp_{\lambda/\mathcal{B}}$, $s_\lambda = o_{\lambda/\mathcal{D}}$. Let us note that various Macdonald identities involve symplectic and orthogonal $S$-functions, which can be associated with identities between matrix elements of standard vertex operators [23].

There are analogues of the Jacobi–Trudi identity, due to Weyl [24], which take the form

$$sp_\lambda = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} + h_{\lambda_1-1} & h_{\lambda_1+2} + h_{\lambda_1-2} & \cdots & h_{\lambda_1+n-1} + h_{\lambda_1-(n-1)} \\ h_{\lambda_2-1} & h_{\lambda_2} + h_{\lambda_2-2} & h_{\lambda_2+1} + h_{\lambda_2-3} & \cdots & h_{\lambda_2+n-2} + h_{\lambda_2-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_n-(n-1)} & h_{\lambda_n-(n-2)} + h_{\lambda_n-n} & h_{\lambda_n-(n-3)} + h_{\lambda_n-(n+1)} & \cdots & h_{\lambda_n} + h_{\lambda_n-(2n-2)} \end{vmatrix} \tag{1.6}$$

and a similar one for $o_\lambda$ but with $h_i \to h_i - h_{i-2}$. There is also a dual identity [25]

$$o_{\lambda'} = \begin{vmatrix} e_{\lambda_1} & e_{\lambda_1+1} + e_{\lambda_1-1} & e_{\lambda_1+2} + e_{\lambda_1-2} & \cdots & e_{\lambda_1+n-1} + e_{\lambda_1-(n-1)} \\ e_{\lambda_2-1} & e_{\lambda_2} + e_{\lambda_2-2} & e_{\lambda_2+1} + e_{\lambda_2-3} & \cdots & e_{\lambda_2+n-2} + e_{\lambda_2-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{\lambda_n-(n-1)} & e_{\lambda_n-(n-2)} + e_{\lambda_n-n} & e_{\lambda_n-(n-3)} + e_{\lambda_n-(n+1)} & \cdots & e_{\lambda_n} + e_{\lambda_n-(2n-2)} \end{vmatrix}$$

and a similar one for $sp_{\lambda'}$ but with $e_i \to e_i - e_{i-2}$. Here $\lambda'$ denotes the conjugate partition of $\lambda$. If $\omega$ denotes the ring homomorphism defined on the power sums by $\omega(p_n) = (-1)^{n-1} p_n$, then it follows from definitions (1.5) that $\omega(sp_\lambda) = o_{\lambda'}$, $\omega(o_\lambda) = sp_{\lambda'}$.

There are also Giambelli formulae (see [26] and references therein) taking the form

$$sp_{(\alpha|\beta)} = \det(sp_{(\alpha_i|\beta_j)}) \qquad o_{(\alpha|\beta)} = \det(o_{(\alpha_i|\beta_j)}). \tag{1.7}$$

In fact, the above determinantal expansions are but the simplest cases of more general expansions in terms of strip decompositions of partitions [20, 27]. From the determinantal expression (1.6), one-part symplectic $S$-functions have the following expansion in terms of normal $S$-functions:

$$\begin{aligned} sp_{(n,1^k)} &= h_n e_k + \sum_{i=1}^{k} (-1)^i (h_{n+i} + h_{n-i}) e_{k-i} \\ &= s_{(n,1^k)} + \sum_{i=1}^{k} (-1)^i (s_{(n-i,1^{k-i})} + s_{(n-i+1,1^{k-i-1})}). \end{aligned} \tag{1.8}$$

Let $R_{ij}$ be a raising operator for partitions, so that for $i < j$,

$$R_{ij}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots) = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots)$$

and extend this to an action on the functions $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$ via $R_{ij} h_\lambda = h_{R_{ij}\lambda}$. Then we have the raising operator formula [28]

$$s_\lambda = \prod_{i<j} (1 - R_{ij}) h_\lambda. \tag{1.9}$$

If we let $L_{ij}$ be a lowering operator for partitions such that for $i \leqslant j$,

$$L_{ij}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots) = (\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_j - 1, \ldots)$$

(where $L_{ii}$ subtracts 2 from the $i$th label) and extend this to $h_\lambda$ as above, then the following identities hold [29]:

$$sp_\lambda = \prod_{i<j} (1 - L_{ij}) \prod_{k<l} (1 - R_{kl}) h_\lambda \tag{1.10}$$

$$o_\lambda = \prod_{i \leqslant j} (1 - L_{ij}) \prod_{k<l} (1 - R_{kl}) h_\lambda. \tag{1.11}$$

There are also Cauchy-type formulae of the form [24, 25]

$$\sum_\lambda sp_\lambda(x) s_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \prod_{k<l} (1 - y_k y_l)$$

$$\sum_\lambda o_\lambda(x) s_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \prod_{k \leqslant l} (1 - y_k y_l).$$

### 1.2. *Vertex operator realization of S-functions*

We shall now briefly review the normal boson–fermion correspondence [1, 30], paying particular attention to the two different ways of generating $S$-functions. This will aid in understanding the symplectic (orthogonal) version of the boson–fermion correspondence which we introduce in the next section. Define vertex operators

$$X(z) = \exp\left( \sum_{n=1}^{\infty} \frac{p_n(x)}{n} z^n \right) \exp\left( - \sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)} z^{-n} \right) \mathrm{e}^{\mathrm{i}q} z^{\alpha_0}$$

$$X^*(z) = \exp\left( - \sum_{n=1}^{\infty} \frac{p_n(x)}{n} z^n \right) \exp\left( \sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)} z^{-n} \right) z^{-\alpha_0} \mathrm{e}^{-\mathrm{i}q}$$

where $p_n(x) = \sum_i x_i^n$ are power sums and the operators $\alpha_0$, $q$ obey $[q, \alpha_0] = \sqrt{-1}$. If one expands these vertex operators in modes

$$X(z) = \sum_{n\in\mathbb{Z}} X_n z^n \qquad X^*(z) = \sum_{n\in\mathbb{Z}} X_n^* z^{-n}$$

then we have for $\lambda = (\lambda_1, \ldots, \lambda_p)$

$$s_\lambda\, \mathrm{e}^{\mathrm{i}pq} = X_{\lambda_1+p-1} X_{\lambda_2+p-2} \cdots X_{\lambda_p} \cdot \mathbf{1} \tag{1.12}$$

$$(-1)^{|\lambda|} s_{\lambda'}\, \mathrm{e}^{-\mathrm{i}pq} = X^*_{-(\lambda_1+p)} X^*_{-(\lambda_2+p-1)} \cdots X^*_{-(\lambda_p+1)} \cdot \mathbf{1} \tag{1.13}$$

while for $\lambda = (i_1, \ldots, i_r | j_1 - 1, \ldots, j_r - 1)$

$$(-1)^{j_1+\cdots j_r} s_\lambda = X^*_{-j_1} \cdots X^*_{-j_r} X_{i_r} \cdots X_{i_1} \cdot \mathbf{1}. \tag{1.14}$$

The reason for the 'momentum' factors in (1.12) and (1.13) is due to their presence in the vertex operators $X(z)$ and $X^*(z)$ (which makes the modes obey free fermionic relations (1.16)). When no confusion arises, we shall drop these momentum factors to make things more readable. To see why (1.12) holds for instance, note that

$$\begin{aligned}
X_{\lambda_1+p-1} X_{\lambda_2+p-2} \cdots X_{\lambda_p} \cdot \mathbf{1} &= \int \frac{\mathrm{d}z}{z} z_1^{-(\lambda_1+p-1)} \cdots z_p^{-\lambda_p} X(z_1) \cdots X(z_p) \cdot \mathbf{1} \\
&= \int \frac{\mathrm{d}z}{z} z_1^{-\lambda_1} \cdots z_p^{-\lambda_p} \prod_{i<j} \left(1 - \frac{z_j}{z_i}\right) \exp\left(\sum_{n=1}^{\infty} \frac{p_n(x)}{n} (z_1^n + \cdots + z_p^n)\right) \cdot \mathrm{e}^{\mathrm{i}pq} \\
&= \int \frac{\mathrm{d}z}{z} z_1^{-\lambda_1} \cdots z_p^{-\lambda_p} \prod_{i<j} \left(1 - \frac{z_j}{z_i}\right) \sum_{i_1,\ldots i_p} h_{i_1} \cdots h_{i_p} z_1^{i_1} \cdots z_p^{i_p} \cdot \mathrm{e}^{\mathrm{i}pq} \\
&= \prod_{i<j} (1 - R_{ij}) h_\lambda \cdot \mathrm{e}^{\mathrm{i}pq}
\end{aligned} \tag{1.15}$$

which in turn is $s_\lambda \cdot \mathrm{e}^{\mathrm{i}pq}$ by (1.9).

However, you can also generate $s_\lambda$ via expectation values of free fermions. Recall [1] that the algebra $\mathcal{A}$ of free fermions is generated by $\psi_i$, $\psi_i^*$, $i \in \mathbb{Z}$ satisfying the anti-commutation relations

$$\{\psi_i, \psi_j\} = 0 = \{\psi_i^*, \psi_j^*\} \qquad \{\psi_i, \psi_j^*\} = \delta_{ij}. \tag{1.16}$$

There is a Fock representation $\mathcal{F}$ of this algebra with a vacuum $|0\rangle$ which satisfies

$$\begin{aligned}
&\psi_i|0\rangle = 0 \quad (i < 0) \qquad &&\psi_i^*|0\rangle = 0 \quad (i \geqslant 0) \\
&\langle 0|\psi_i = 0 \quad (i \geqslant 0) \qquad &&\langle 0|\psi_i^* = 0 \quad (i < 0).
\end{aligned} \tag{1.17}$$

The following states are $gl(\infty)$ highest weight states (which, as we shall later see, correspond to the states $\mathrm{e}^{\mathrm{i}pq}\mathbf{1}$)

$$|p\rangle = \begin{cases} \psi_{p-1} \cdots \psi_0 |0\rangle & p > 0 \\ |0\rangle & p = 0 \\ \psi_p^* \cdots \psi_{-1}^* |0\rangle & p < 0. \end{cases} \tag{1.18}$$

The corresponding dual states take the form

$$\langle p| = \begin{cases} \langle 0|\psi_0^* \cdots \psi_{p-1}^* & p > 0 \\ \langle 0| & p = 0 \\ \langle 0|\psi_{-1} \cdots \psi_p & p < 0. \end{cases} \tag{1.19}$$

Using products of free fermions, we can define generators $H_n = \sum_{i\in\mathbb{Z}} \psi_i \psi^*_{i+n}$, $n \neq 0$, which obey [1]

$$[H_n, H_m] = n\delta_{n+m,0} \qquad [H_n, \psi_i] = \psi_{i-n} \qquad [H_n, \psi_i^*] = -\psi^*_{i+n}.$$

Finally, define the Hamiltonian $H(x) = \sum_{j\geqslant 1}(1/j)p_j(x)H_j$. Then using Wick's theorem, it is seen that the $S$-function $s_\lambda$ can be constructed from the following expectation values:

$$s_\lambda = \langle p|\mathrm{e}^{H(x)}\psi_{\lambda_1+p-1}\psi_{\lambda_2+p-2}\cdots\psi_{\lambda_p}|0\rangle \tag{1.20}$$

$$(-1)^{|\lambda|}s_{\lambda'} = \langle -p|\,\mathrm{e}^{H(x)}\psi^*_{-(\lambda_1+p)}\psi^*_{-(\lambda_2+p-1)}\cdots\psi^*_{-(\lambda_p+1)}|0\rangle \tag{1.21}$$

and

$$(-1)^{j_1+\cdots j_r}s_\lambda = \langle 0|\,\mathrm{e}^{H(x)}\psi^*_{-j_1}\cdots\psi^*_{-j_r}\psi_{i_1}\cdots\psi_{i_r}|0\rangle. \tag{1.22}$$

To see why for example, (1.12) and (1.20) are the same, first note that, using

$$\mathrm{e}^{H(x)}\psi(z)\,\mathrm{e}^{-H(x)} = \exp\left(\sum_n p_n(x)z^n/n\right)\psi(z)$$

the expectation value on the right-hand side of (1.20) can be rewritten as

$$\int \frac{\mathrm{d}z}{z} z_1^{-(\lambda_1+p-1)}\cdots z_p^{-\lambda_p}\langle p|\psi(z_1)\cdots\psi(z_p)|0\rangle \exp\left(\sum_{n=1}^{\infty}\frac{p_n(x)}{n}(z_1^n+\cdots+z_p^n)\right)$$

but since [1]

$$\langle p|\psi(z_1)\cdots\psi(z_p)|0\rangle = \prod_{i<j}(z_i - z_j) \tag{1.23}$$

we get exactly the same factors that appear in (1.15). So we can produce $S$-functions either by considering the action of the modes $X_n$, $X_n^*$ on $\mathbf{1}$, or by computing the expectation value of free fermions $\psi_i$, $\psi_j^*$ with a time evolution like the Hamiltonian $\exp(H(x))$. The normal ordering of the vertex operators $X(z)$ and $X^*(z)$ generates the same factors as the 'correlation function' (1.23). The rationale for explaining the above in great detail is that we shall generate, for example, the symplectic $S$-functions as the action of the modes of certain vertex operators $\Gamma(z)$ , $\Gamma^*(z)$ on $\mathbf{1}$. We will then show that you can also consider this as *either* (i) the expectation of free fermions with a 'modified' Hamiltonian $\exp(H_A(x))$ *or* (ii) the expectation of free fermions with the ordinary Hamiltonian $\exp(H(x))$, but with a 'modified' bra vacuum $\langle 0|'$. It is this latter viewpoint which will allow a bilinear identity to be generated, and hence a hierarchy of Hirota equations.

## 2. Vertex operator realization of symplectic and orthogonal $S$-functions

First note that we can write the series $\mathcal{A}(x)$ (1.1), in the form

$$\mathcal{A}(x) = \exp\left(\sum_{j\geqslant 1}\frac{1}{2j}(p_{2j} - p_j^2)\right).$$

If $D$ (not to be confused with the set $D$ of partitions defined after (1.4)) denotes the adjoint of multiplication under the normal $S$-function inner product (i.e. $D(s_\mu)s_\lambda = s_{\lambda/\mu}$) then, due to the fact that $D(a)D(b) = D(ab)$, we have

$$\begin{aligned} sp_\lambda = s_{\lambda/\mathcal{A}} &= D(\mathcal{A})s_\lambda \\ &= \exp\left(\sum_{j\geqslant 1}\frac{1}{2j}(D(p_{2j}) - D^2(p_j))\right)s_\lambda \\ &= \exp\left(\sum_{j\geqslant 1}\left(\frac{\partial}{\partial p_{2j}} - \frac{j}{2}\frac{\partial}{\partial p_j^2}\right)\right)s_\lambda. \end{aligned}$$

Suppose we now express the ordinary $S$-function $s_\lambda$ in the form of modes $X_n$ acting on $\mathbf{1}$. What does the corresponding symplectic function $sp_\lambda$ then look like when we transfer the above exponential operator through the modes $X_n$ to act on $\mathbf{1}$? First of all, we are going to need the next term in the Baker–Cambell–Hansdorff formula, namely

$$e^A\, e^B\, e^{-A} = e^{[A,B]+[[A,B],B]/2+\cdots}.$$

Thus, letting

$$e^K = \exp\left(\sum_{j\geqslant 1}\left(\frac{\partial}{\partial p_{2j}} - \frac{j}{2}\frac{\partial}{\partial p_j^2}\right)\right)$$

we see that

$$e^K X(z)\, e^{-K} = X(z)\exp\left(-\sum_{j\geqslant 1}\frac{\partial}{\partial p_j}z^j\right) \tag{2.1}$$

$$e^K X^*(z)\, e^{-K} = (1-z^2)X^*(z)\exp\left(\sum_{j\geqslant 1}\frac{\partial}{\partial p_j}z^j\right). \tag{2.2}$$

So, in terms of the modes $X_n$, the symplectic functions take the form (ignoring the momentum factor $e^{ipq}$ for the moment)

$$\begin{aligned} sp_\lambda &= \int \frac{dz}{z} z_1^{-(\lambda_1+p-1)}\cdots z_p^{-\lambda_p}\, e^K X(z_1)\cdots X(z_p)\cdot\mathbf{1} \\ &= \int \frac{dz}{z} z_1^{-(\lambda_1+p-1)}\cdots z_p^{-\lambda_p}\prod_{i<j}(1-z_iz_j)X(z_1)\cdots X(z_p)\cdot\mathbf{1} \end{aligned} \tag{2.3}$$

where the factors $(1-z_iz_j)$ arise from shuffling the extraneous differential operators appearing in (2.1) through the operators $X(z_1),\ldots,X(z_p)$. Thus we have

$$sp_\lambda = \prod_{i<j}(1-L_{ij})s_\lambda = \prod_{i<j}(1-L_{ij})(1-R_{ij})h_\lambda$$

recovering (1.10). In a similar fashion, through using (2.2), one obtains the 'dual' expression

$$\begin{aligned} (-1)^{|\lambda|}sp_{\lambda'} &= \int \frac{dz}{z} z_1^{-(\lambda_1+p)}\cdots z_p^{-(\lambda_p+1)}\, e^{D(A)} X^*(z_1)X^*(z_2)\cdots X^*(z_p)\cdot\mathbf{1} \\ &= \int \frac{dz}{z} z_1^{-(\lambda_1+p)}\cdots z_p^{-(\lambda_p+1)}\prod_{i\leqslant j}(1-z_iz_j)X^*(z_1)X^*(z_2)\cdots X^*(z_p)\cdot\mathbf{1} \end{aligned}$$

so that $sp_{\lambda'} = \prod_{i\leqslant j}(1-L_{ij})s_{\lambda'}$. By this, we mean that $L_{ij}s_{\lambda'} = s_{(L_{ij}\lambda)'}$, so that, for example, $L_{12}s_{(2^41)} = s_{(2^21)}$.

Turning to the orthogonal case, and writing the series $\mathcal{C}(x)$ appearing in (1.3) as

$$\mathcal{C}(x) = \exp\left(-\sum_{j\geqslant 1}\frac{1}{2j}(p_{2j}+p_j^2)\right)$$

we see that

$$o_\lambda = \int \frac{dz}{z} z_1^{-(\lambda_1+p-1)}\cdots z_p^{-\lambda_p}\prod_{i\leqslant j}(1-z_iz_j)X(z_1)\cdots X(z_p)\cdot\mathbf{1}$$

which implies (1.11)

$$o_\lambda = \prod_{i\leqslant j}(1-L_{ij})s_\lambda = \prod_{i\leqslant j}(1-L_{ij})\prod_{k<l}(1-R_{kl})h_\lambda.$$

A similar calculation gives $o_{\lambda'} = \prod_{i<j}(1-L_{ij})s_{\lambda'}$.

Now, if you take a close look at expression (2.3) for $sp_\lambda$, and normal order the $X(z)$ VOs appearing there, you have in fact

$$sp_\lambda = \int \frac{\mathrm{d}z}{z} z_1^{-\lambda_1} \cdots z_p^{-\lambda_p} \prod_{i<j}(1 - z_i z_j)\left(1 - \frac{z_j}{z_i}\right) \exp\left(\sum_{n=1}^{\infty} \frac{p_n(x)}{n}(z_1^n + \cdots + z_p^n)\right). \tag{2.4}$$

Let us seek to achieve the same expression as this one by applying the modes of some 'modified' VO to **1**. Define

$$\Gamma(z) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(x)}{n} z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)}(z^{-n} + z^n)\right) \mathrm{e}^{\mathrm{i}q} z^{\alpha_0}$$
$$\Gamma^*(z) = (1 - z^2) \exp\left(-\sum_{n=1}^{\infty} \frac{p_n(x)}{n} z^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)}(z^{-n} + z^n)\right) z^{-\alpha_0}\,\mathrm{e}^{-\mathrm{i}q}. \tag{2.5}$$

Then from the two VO normal-order relation

$$\begin{aligned}\Gamma(z)\Gamma(w) &= (z - w)(1 - zw) : \Gamma(z)\Gamma(w) : \\ &= (z - w)(1 - zw) \exp\left(\sum_{n=1}^{\infty} \frac{p_n(x)}{n}(z^n + w^n)\right) \\ &\quad \times \exp\left(-\sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)}(z^{-n} + z^n + w^{-n} + w^n)\right) \mathrm{e}^{2\mathrm{i}q}(zw)^{\alpha_0}\end{aligned} \tag{2.6}$$

or, to be more precise, its $p$ VO generalization, you see that (again ignoring the momentum factor $\mathrm{e}^{\mathrm{i}pq}$ which should appear on the left-hand side)

$$sp_\lambda = \int \frac{\mathrm{d}z}{z} z_1^{-(\lambda_1+p-1)} \cdots z_p^{-\lambda_p} \Gamma(z_1) \cdots \Gamma(z_p) \cdot \mathbf{1}.$$

Hence, if the vertex operator $\Gamma(z)$ has the mode expansion $\Gamma(z) = \sum_{n\in\mathbb{Z}} \Gamma_n z^n$, we have the result

$$sp_\lambda = \Gamma_{\lambda_1+p-1}\Gamma_{\lambda_2+p-2} \cdots \Gamma_{\lambda_p} \cdot \mathbf{1}. \tag{2.7}$$

The same consideration applies for the dual and Giambelli forms, so if $\Gamma^*(z)$ has the mode expansion $\Gamma^*(z) = \sum_{n\in\mathbb{Z}} \Gamma_n^* z^{-n}$ then

$$(-1)^{|\lambda|} sp_{\lambda'} = \Gamma^*_{-(\lambda_1+p)}\Gamma^*_{-(\lambda_2+p-1)} \cdots \Gamma^*_{-(\lambda_p+1)} \cdot \mathbf{1} \tag{2.8}$$
$$(-1)^{j_1+\cdots j_r} sp_\lambda = \Gamma^*_{-j_1} \cdots \Gamma^*_{-j_r} \Gamma_{i_r} \cdots \Gamma_{i_1} \cdot \mathbf{1}. \tag{2.9}$$

Now we know that the modes $X_i$, $X_j^*$ obey free fermionic anti-commutation relations. What about the modes $\Gamma_i$ and $\Gamma_j^*$? It turns out that they *also* obey free fermionic relations. The calculations which show this are almost the same as for the $S$-function case. From the antisymmetry between $z$ and $w$ in equation (2.6), it is clear that $\Gamma(z)\Gamma(w) + \Gamma(w)\Gamma(z) = 0$. Thus, in terms of modes, $\{\Gamma_i, \Gamma_j\} = 0$. Similarly, we have

$$\begin{aligned}\Gamma^*(z)\Gamma^*(w) &= (1 - z^2)(1 - w^2)(w^{-1} - z^{-1})(1 - zw) \exp\left(-\sum_{n=1}^{\infty} \frac{p_n(x)}{n}(z^n + w^n)\right) \\ &\quad \times \exp\left(\sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)}(z^{-n} + z^n + w^{-n} + w^n)\right)(zw)^{-\alpha_0}\,\mathrm{e}^{-2\mathrm{i}q}\end{aligned} \tag{2.10}$$

which is also antisymmetric in $z$ and $w$, so that $\{\Gamma_i^*, \Gamma_j^*\} = 0$. Finally, from the relations

$$\Gamma(z)\Gamma^*(w) = (1-w^2)(1-zw)^{-1}\frac{w/z}{(1-w/z)}\exp\left(\sum_{n=1}^{\infty}\frac{p_n(x)}{n}(z^n-w^n)\right)$$
$$\Gamma^*(w)\Gamma(z) = (1-w^2)(1-zw)^{-1}\frac{1}{(1-z/w)}\exp\left(\sum_{n=1}^{\infty}\frac{p_n(x)}{n}(z^n-w^n)\right)$$

and the fact that

$$\frac{w/z}{(1-w/z)} + \frac{1}{(1-z/w)} = \sum_{n\in\mathbb{Z}}\left(\frac{w}{z}\right)^n = \delta\left(\frac{w}{z}\right)$$

we have

$$\{\Gamma(z), \Gamma^*(w)\} = (1-w^2)(1-zw)^{-1}\exp\left(\sum_{n=1}^{\infty}\frac{p_n(x)}{n}(z^n-w^n)\right)\delta\left(\frac{w}{z}\right) = \delta\left(\frac{w}{z}\right)$$

where we have used the standard trick of noting that for any function $f$, we have $f(w)\delta(w/z) = f(z)\delta(w/z)$. Thus $\{\Gamma_i, \Gamma_j^*\} = \delta_{ij}$. So, the modes $\Gamma_i$, $\Gamma_j^*$ obey free fermionic relations just like the modes $X_i$, $X_j^*$. This then begs the question: can a symplectic $S$-function be generated as the expectation value of a Hamiltonian and free fermions $\psi_i$, $\psi_j^*$ as can be done for the $S$-function case (see (1.22))? The answer is yes, but their are two ways you can view it, as we shall see in the next section.

Orthogonal functions can be treated in exactly the same way, so that if

$$\Omega(z) = (1-z^2)\exp\left(\sum_{n=1}^{\infty}\frac{p_n(x)}{n}z^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{\partial}{\partial p_n(x)}(z^{-n}+z^n)\right)\mathrm{e}^{\mathrm{i}q}z^{\alpha_0}$$
$$\Omega^*(z) = \exp\left(-\sum_{n=1}^{\infty}\frac{p_n(x)}{n}z^n\right)\exp\left(\sum_{n=1}^{\infty}\frac{\partial}{\partial p_n(x)}(z^{-n}+z^n)\right)z^{-\alpha_0}\mathrm{e}^{-\mathrm{i}q}$$

have modal expansions $\Omega(z) = \sum_{n\in\mathbb{Z}}\Omega_n z^n$, $\Omega^*(z) = \sum_{n\in\mathbb{Z}}\Omega_n^* z^{-n}$, then

$$o_\lambda = \Omega_{\lambda_1+p-1}\Omega_{\lambda_2+p-2}\cdots\Omega_{\lambda_p}\cdot\mathbf{1}$$
$$(-1)^{|\lambda|}o_{\lambda'} = \Omega^*_{-(\lambda_1+p)}\Omega^*_{-(\lambda_2+p-1)}\cdots\Omega^*_{-(\lambda_p+1)}\cdot\mathbf{1}$$
$$(-1)^{j_1+\cdots j_r}o_\lambda = \Omega^*_{-j_1}\cdots\Omega^*_{-j_r}\Omega_{i_r}\cdots\Omega_{i_1}\cdot\mathbf{1}.$$

The modes $\Omega_i$, $\Omega_j^*$ also obey free fermionic relations. Indeed, it can be seen that they are related to the modes $\Gamma_i$, $\Gamma_j^*$ by

$$\Omega_n = \Gamma_n - \Gamma_{n-2} \qquad \Omega_n^* = \sum_{i\geqslant 0}\Gamma^*_{n+2i}$$

or equivalently

$$\Gamma_n = \sum_{i\geqslant 0}\Omega_{n-2i} \qquad \Gamma_n^* = \Omega_n^* - \Omega_{n+2}^*.$$

From now on, we restrict our attention to the symplectic case, as the orthogonal case can be treated in an identical fashion.

## 3. Approach 1: modified Hamiltonian

Returning to the question of whether symplectic $S$-functions can be constructed from an expectation value, the first way is to consider the expectation value of the free fermions $\psi_i$, $\psi_j^*$, with the modified time evolution operator $\exp(H_A(x))$ but with the standard vacua $\langle 0|$, and $|0\rangle$ which obey (1.17). The necessary Hamiltonian is (as the reader might have already guessed)

$$H_A(x) = \sum_{j\geqslant 1} \frac{1}{j}(p_j(x)H_j + \tfrac{1}{2}H_{2j} - \tfrac{1}{2}H_j^2).$$

Why is this the right Hamiltonian? Because the time evolution of the fermionic currents $\psi(z)$, $\psi^*(z)$ generates the correct terms to mimic the mode calculations we performed in the previous section. For example, you can check that

$$\exp(H_A(x))\psi(z)\exp(-H_A(x)) = \exp\left(\sum_{j\geqslant 1}\frac{p_j(x)}{j}z^j\right)\psi(z)\exp\left(-\sum_{j\geqslant 1}\frac{1}{j}H_j z^j\right) \qquad (3.1)$$

$$\exp(H_A(x))\psi^*(z)\exp(-H_A(x)) = (1-z^2)\exp\left(-\sum_{j\geqslant 1}\frac{p_j(x)}{j}z^j\right)\psi^*(z)$$
$$\times\exp\left(\sum_{j\geqslant 1}\frac{H_j}{j}z^j\right). \qquad (3.2)$$

Thus if you look at the expectation value

$$\langle p|\,\mathrm{e}^{H_A(x)}\psi_{\lambda_1+p-1}\psi_{\lambda_2+p-2}\cdots\psi_{\lambda_p}|0\rangle = \int\frac{\mathrm{d}z}{z}z_1^{-(\lambda_1+p-1)}\cdots z_p^{-\lambda_p}\langle p|\,\mathrm{e}^{H_A(x)}\psi(z_1)\cdots\psi(z_p)|0\rangle$$

insert factors $\exp(H_A(x))\exp(-H_A(x))$ everywhere, and then use (3.1), then the above expectation value takes the form

$$\int\frac{\mathrm{d}z}{z}z_1^{-(\lambda_1+p-1)}\cdots z_p^{-\lambda_p}\langle p|\psi(z_1)\exp\left(-\sum_{j\geqslant 1}\frac{1}{j}H_j z_1^j\right)\psi(z_2)\exp\left(-\sum_{j\geqslant 1}\frac{1}{j}H_j z_2^j\right)\cdots$$
$$\times\exp\left(-\sum_{j\geqslant 1}\frac{1}{j}H_j z_{p-1}^j\right)\psi(z_p)|0\rangle\exp\left(\sum_{j\geqslant 1}\frac{p_j(x)}{j}(z_1^j+\cdots+z_p^j)\right).$$

You then shuffle the remaining exponentials away using the (normal) time evolution

$$\exp\left(-\sum_{j\geqslant 1}\frac{1}{j}H_j z^j\right)\psi(w)\exp\left(\sum_{j\geqslant 1}\frac{1}{j}H_j z^j\right) = (1-zw)\psi(z)$$

to recover

$$\int\frac{\mathrm{d}z}{z}z_1^{-(\lambda_1+p-1)}\cdots z_p^{-\lambda_p}\prod_{i<j}(1-z_iz_j)\langle p|\psi(z_1)\cdots\psi(z_p)|0\rangle$$
$$\times\exp\left(\sum_{j\geqslant 1}\frac{p_j(x)}{j}(z_1^j+\cdots+z_p^j)\right)$$
$$= \int\frac{\mathrm{d}z}{z}z_1^{-\lambda_1}\cdots z_p^{-\lambda_p}\prod_{i<j}(1-z_iz_j)\left(1-\frac{z_j}{z_i}\right)$$
$$\times\exp\left(\sum_{j\geqslant 1}\frac{p_j(x)}{j}(z_1^n+\cdots+z_p^n)\right) \qquad \text{by (1.23)}$$
$$= sp_\lambda \qquad \text{by (2.4).}$$

Similar considerations apply for the dual and Giambelli forms for $sp_\lambda$. Although this modified Hamiltonian $H_A(x)$ allow symplectic $S$-functions to be expressed as expectation values of free fermions, one important disadvantage of this formulation is that Wick's theorem no longer applies, for the simple reason that the time evolutions (3.1) and (3.2) are no longer linear in free fermions, like the normal case (recall that the operators $H_n$ are bilinear in $\psi_i$, $\psi_i^*$). Nevertheless, one can still get the known determinantal forms of $sp_\lambda$ either from the modal expressions, or from the expectation values shown above (see the appendix for an example).

## 4. Approach 2: modified vacuum

The second approach is to leave the Hamiltonian alone and consider some sort of modified fermionic vacuum, in the sense that some of the equations in (1.17) are no longer valid.

### *4.1. S-function case*

First, we shall discuss how the fermionic inner product and 'modal' ($S$-function) inner product are related. Given states of the form $a|0\rangle$, $b|0\rangle$ where $a, b \in \mathcal{A}$, the inner product of these states is just given by $(a|0\rangle, b|0\rangle) = \langle 0|a^* b|0\rangle$. Turning to the modes, define an operator $\mathbf{1}^*$ (the dual vacuum) so that its action on the function $\mathbf{1}$ is just $\mathbf{1}^* \cdot \mathbf{1} = 1$. Then the inner product of states of the form $A \cdot \mathbf{1}$, $B \cdot \mathbf{1}$, where $A$ and $B$ are composed of the modes $X_i$, $X_j^*$ is given by $(A \cdot \mathbf{1}, B \cdot \mathbf{1}) = (\mathbf{1}, A^* B \cdot \mathbf{1}) = \mathbf{1}^* A^* B \cdot \mathbf{1}$. Note that in the symmetric function sense, $X^*(z)$ really *is* the adjoint operator of $X(z)$, because the ajoint of $p_n$ is just $D(p_n) \equiv n\partial/\partial p_n$ and *vice versa* (one also must define $(\alpha_0)^* = -\alpha_0$, and $(q)^* = -q$). This inner product thus reproduces the standard $S$-function inner product under which $(s_\lambda, s_\mu) = \delta_{\lambda\mu}$. We shall return to this later.

Now let us clarify the relationship between the fermionic vacuum $|0\rangle$ and the $gl(\infty)$ highest weight states $|p\rangle$ and their modal counterparts 1 and $\mathrm{e}^{\mathrm{i}pq}$ for the normal BFC. Recall the definition (1.18) of the kets $|p\rangle$. By replacing the $\psi_i$'s in the above definitions with $X_i$'s, and $|0\rangle$ with $\mathbf{1}$, one recovers the 'state' $\mathrm{e}^{\mathrm{i}pq} \cdot \mathbf{1}$, which can be seen by setting $\lambda = 0$ in (1.12) and (1.13) for the cases $p > 0$ and $p < 0$, respectively. Also, from definition (1.19) of the bras $\langle p|$, the replacement of the $\psi_i$'s with $X_i$'s here (along with replacing $\langle 0|$ with $\mathbf{1}^*$) yields the dual 'states' $\mathbf{1}^* \cdot \mathrm{e}^{-\mathrm{i}pq}$. To see this, take $p > 0$, so that

$$
\begin{aligned}
\mathbf{1}^* \cdot X_0^* \cdots X_{p-1}^* &= \mathbf{1}^* \cdot \int \frac{\mathrm{d}z}{z} z_1^0 z_2^1 \cdots z_p^{p-1} X^*(z_1) X^*(z_2) \cdots X^*(z_p) \\
&= \mathbf{1}^* \cdot \mathrm{e}^{-\mathrm{i}pq} \int \frac{\mathrm{d}z}{z} \prod_{i<j} \left(1 - \frac{z_j}{z_i}\right) \exp\left(\sum_{j\geqslant 1} \frac{p_j(x)}{j}(z_1^j + \cdots + z_p^j)\right) \\
&= \mathbf{1}^* \cdot \mathrm{e}^{-\mathrm{i}pq}.
\end{aligned}
\tag{4.1}
$$

Here, we are using the property that the dual vacuum $\mathbf{1}^*$ is killed by power sums $p_n(x)$, just as the vacuum $\mathbf{1}$ is killed by $\partial/\partial p_n(x)$.

We have seen how the states $\langle p|$ and $|p\rangle$ are formed in the 'modal' language. Let us now see how the modal vacua $\mathbf{1}$ and $\mathbf{1}^*$ are annihilated by the modes $X_j$, $X_j^*$. Indeed, we have the equivalent of (1.17) of the form

$$
\begin{aligned}
&X_j \cdot \mathbf{1} = 0 \quad (j < 0) \qquad && X_j^* \cdot \mathbf{1} = 0 \quad (j \geqslant 0) \\
&\mathbf{1}^* \cdot X_j = 0 \quad (j \geqslant 0) \qquad && \mathbf{1}^* \cdot X_j^* = 0 \quad (j < 0).
\end{aligned}
\tag{4.2}
$$

To see this, we have for example

$$\mathbf{1}^* \cdot X_j = \int \frac{\mathrm{d}z}{z} z^{-j} X(z) = \mathrm{e}^{\mathrm{i}q} \cdot \int \frac{\mathrm{d}z}{z} z^{-j-1} \sum_{n\geqslant 0} D(e_n)(-z)^{-n} = 0 \qquad \text{if } j \geqslant 0.$$

When we turn to the symplectic case, we will see that (4.2) no longer holds, and this is why it looks like the vacuum is 'modified'.

### *4.2. Symplectic S-function case*

First, let us take a look at the inner product between states of the form $A \cdot \mathbf{1}$ where $A$ is composed of modes $\Gamma_i$, $\Gamma_i^*$. Define it in the normal way: $(A \cdot \mathbf{1}, B \cdot \mathbf{1})' = (\mathbf{1}, A^* B \cdot \mathbf{1})' = \mathbf{1}^* A^* B \cdot \mathbf{1}$ where $A^*$ is formed from $A$ by reversing the order of all the modes and interchanging $\Gamma_i \leftrightarrow \Gamma_i^*$. Due to the fact that the modes $\Gamma_i$, $\Gamma_i^*$ obey free fermion relations, all the different states in (2.7), (2.8), or (2.9) are clearly orthonormal. Thus we have a new inner product on the ring of symmetric functions under which symplectic $S$-functions are orthonormal. However, this is not a particularly nice inner product for several reasons. First, by looking at the expressions for the vertex operators $\Gamma(z)$, and $\Gamma^*(z)$ in (2.5), $\Gamma^*(z)$ is not the adjoint of $\Gamma(z)$ in the normal symmetric function sense (with $D(p_n)$ being the adjoint of $p_n$). Thus the adjoints of symmetric functions with respect to the inner product $(\cdot,\cdot)'$ are no longer the normal ones. The inner product $(\cdot,\cdot)'$ is still positive definite for the normal $S$-functions i.e. $(s_\lambda, s_\lambda)' > 0$ $\forall\lambda$. Second, it exhibits pathological behaviour in the sense that there exists a state $D \cdot \mathbf{1}$ which represents the symmetric function 0, and a state $E \cdot \mathbf{1}$ which represents a non-zero symmetric function, such that $(D \cdot \mathbf{1}, E \cdot \mathbf{1})' \neq 0$.

Taking a look at the properties of the vacua $\mathbf{1}$ and $\mathbf{1}^*$, note that although (4.2) is still satisfied with the modes $\Gamma_i$, $\Gamma_i^*$, (4.2) is not. This is clear from the appearance of the term $z^n$, $n > 0$ in the annihilation part of the vertex operators $\Gamma(z)$, $\Gamma^*(z)$. Let us look at an example which will exhibit this property, and the strange behaviour of $(\cdot,\cdot)'$ discussed above. We have the following vacuum expectation value for $j \geqslant 0$:

$$\begin{aligned} \langle \Gamma_i^* \Gamma_j \rangle := (\Gamma_i \cdot \mathbf{1}, \Gamma_j \cdot \mathbf{1})' &= \int \frac{\mathrm{d}z\,\mathrm{d}w}{zw} z^{-i} w^j \mathbf{1}^* \cdot \Gamma^*(z)\Gamma(w) \cdot \mathbf{1} \\ &= \int \frac{\mathrm{d}z\,\mathrm{d}w}{zw} z^{-i} w^j (1-z^2)(1-zw)^{-1}(1-w/z)^{-1} = \delta_{ij} - \delta_{i,-j-2}. \end{aligned} \tag{4.3}$$

Thus for example $(\Gamma_{-2-j} \cdot \mathbf{1}, \Gamma_j \cdot \mathbf{1})' = -1$. In terms of symplectic $S$-functions, for $j \geqslant 0$, $\Gamma_j \cdot \mathbf{1}$ is just $sp_{(n)} = h_n$, but $\Gamma_{-2-j} \cdot \mathbf{1}$ is zero (since (4.2) still holds) giving a prime example of the bad behaviour of the inner product $(\cdot,\cdot)'$.

What about modal representations of the states $\mathrm{e}^{\mathrm{i}pq} \cdot \mathbf{1}$ and $\mathbf{1}^* \cdot \mathrm{e}^{-\mathrm{i}pq}$? A calculation such as that leading to (4.1) shows that

$$\mathrm{e}^{\mathrm{i}pq} \cdot \mathbf{1} = \begin{cases} \Gamma_{p-1} \cdots \Gamma_0 |0\rangle & p > 0 \\ |0\rangle & p = 0 \\ \Gamma_p^* \cdots \Gamma_{-1}^* |0\rangle & p < 0. \end{cases}$$

However, $\mathbf{1}^* \cdot \mathrm{e}^{-\mathrm{i}pq}$ has no equivalent representation to (1.19), due to the presence of the $z^n$ in the differential part of $\Gamma(z)$, and $\Gamma^*(z)$.

Let us now turn back to the free fermions $\psi_i$, $\psi_i^*$, and the bra and kets $\langle 0|$, $|0\rangle$ and see how the above 'modal' formalism can be translated back into the language of free fermions. The fundamental difference we need in the symplectic case is to replace the bra $\langle 0|$ with a new bra $\langle 0|'$ so that, instead of (1.17), we only have

$$\psi_i |0\rangle = 0 \quad (i < 0) \qquad \psi_i^* |0\rangle = 0 \quad (i \geqslant 0).$$

There are, however, some special conditions on $\langle 0|'$ (or equivalently $\mathbf{1}^*$) so that instead of $\langle \psi_i^* \psi_j \rangle = \delta_{ij}$, $j \geqslant 0$ we have (cf (4.3)) $\langle \psi_i^* \psi_j \rangle' = \delta_{ij} - \delta_{i,-j-2}$ or in terms of currents, instead of $\langle \psi^*(z)\psi(w)\rangle = (1-w/z)^{-1}$, we have

$$\langle \psi^*(z)\psi(w)\rangle' := \langle 0|'\psi^*(z)\psi(w)|0\rangle = (1-z^2)(1-zw)^{-1}(1-w/z)^{-1}.$$

This comes about because

$$\langle p|'(\psi_j^* + \psi_{-j-2+2p}^*) = 0 \tag{4.4}$$

(or equivalently, because it can be seen that $\mathbf{1}^* \cdot \mathrm{e}^{-\mathrm{i}pq}$ is annihilated on the right by $\Gamma^*(z) + z^{-2p+2}\Gamma^*(z^{-1})$) so that, for example, for $j > 0$, $\langle \psi_{-j-2}^* \psi_j^* \rangle' = -\langle \psi_j^* \psi_j^* \rangle' = -1$. Similarly, $\mathbf{1}^* \cdot \mathrm{e}^{-\mathrm{i}pq}$ is annihilated by $\Gamma(z) - z^{-2+2p}\Gamma(z^{-1})$ so that

$$\langle p|'(\psi_j - \psi_{-j-2+2p}) = 0. \tag{4.5}$$

The conditions (4.4) and (4.5) are enough to ensure that the expectation values of currents $\psi(z)$, $\psi^*(z)$ are the same as their modal counterparts, for example for $p > 0$,

$$\begin{aligned}\langle p|'\psi(z_1)\cdots\psi(z_{m+p})\psi^*(w_1)\cdots\psi^*(w_m)|0\rangle &= z_1^{p-1}\cdots z_{m+p}^{-m} w_1^m \cdots w_m \prod_i (1-w_i^2) \\ &\quad \times \frac{\prod_{i<j}(1-z_iz_j)(1-z_j/z_i)\prod_{k<l}(1-w_kw_l)(1-w_k/w_l)}{\prod_{i,j}(1-z_iw_j)(1-w_j/z_i)}.\end{aligned} \tag{4.6}$$

From this and similar formulae, it follows that the symplectic $S$-functions can be expressed as the 'modified' expectation value of a string of $\psi_i$'s with the *normal* Hamiltonian $H(x)$, so that (cf (1.20))

$$sp_\lambda = \langle p|' \,\mathrm{e}^{H(x)} \psi_{\lambda_1+p-1}\psi_{\lambda_2+p-2}\cdots\psi_{\lambda_p}|0\rangle$$

and so on.

## 5. Applications

As an application of the above VO construction of the symplectic and orthogonal $S$-functions, we shall show how multiplication and plethysms of these types of symmetric functions can be calculated in the spirit of [31, 32]. We provide details for the symplectic case only, the orthogonal case being almost identical.

### *5.1. Multiplication*

Multiplication of symplectic and orthogonal $S$-functions can be carried out using the Newell–Littlewood rules [33, 34] (for a nice proof see [21])

$$\begin{aligned} sp_\lambda sp_\mu &= \sum_\xi sp_{(\lambda/\xi)\cdot(\mu/\xi)} = \sum_{\xi\alpha\beta\tau} c_{\xi\alpha}^{\lambda} c_{\xi\beta}^{\mu} c_{\alpha\beta}^{\tau} sp_\tau \\ o_\lambda o_\mu &= \sum_\xi o_{(\lambda/\xi)\cdot(\mu/\xi)} = \sum_{\xi\alpha\beta\tau} c_{\xi\alpha}^{\lambda} c_{\xi\beta}^{\mu} c_{\alpha\beta}^{\tau} o_\tau. \end{aligned} \tag{5.1}$$

Alternatively, we can use the vertex operator formalism to express the product of two symplectic functions in terms of symplectic functions associated with non-standard partitions which can be modified according to the usual rules (i.e. the same as the rules for non-standard $S$-functions).

Note that the generating function for $r$ part symplectic $S$-functions $sp_{(n_1,n_2,\ldots,n_r)}$ is given by

$$\begin{aligned}
R(z_1,\ldots,z_r) &= \sum_{n_1,\ldots,n_r\geqslant 0} sp_{(n_1,\ldots,n_r)} z_1^{n_1}\cdots z_r^{n_r} \\
&= \prod_{i<j}(1-z_iz_j)(1-z_j/z_i)\exp\left(\sum_{k\geqslant 1}\frac{p_k}{k}(z_1^k+\cdots+z_r^k)\right) \\
&= \Gamma(z_1)\cdots\Gamma(z_r)\eta(z_1,\ldots,z_r)
\end{aligned}$$

where

$$\eta(z_1,\ldots,z_r) = \exp\left(\sum_{k\geqslant 1}\frac{\partial}{\partial p_k}(z_1^k+z_1^{-k}+\cdots+z_r^k+z_r^{-k})\right) z_1^{-\alpha_0-r+1}\cdots z_r^{-\alpha_0}\,\mathrm{e}^{-riq}.$$

Upon using the relation

$$\eta(z_1,\ldots,z_r)\Gamma(w) = \prod_{i=1}^r \frac{w/z_i}{(1-wz_i)(1-w/z_i)}\Gamma(w)\eta(z_1,\ldots,z_r)$$

we see that

$$sp_{(n_1,\ldots,n_r)}sp_{(m_1,\ldots,m_p)} = \sum_{\{a_{ij}\},\{b_{ij}\}} sp_{\lambda(a;b)}$$

where $\lambda(a;b) = (\lambda_1,\ldots,\lambda_r,\lambda_{r+1},\ldots,\lambda_{r+p})$ with

$$\lambda_i = n_i + \sum_{k=1}^p (b_{ik}-a_{ik}) \qquad 1\leqslant i\leqslant r$$

$$\lambda_{r+i} = m_i - \sum_{k=1}^r (b_{ki}+a_{ki}) \qquad 1\leqslant i\leqslant p.$$

The indices $a_{ij}$, $b_{ij}$, $1\leqslant i\leqslant r$, $1\leqslant j\leqslant p$, are subject to the constraints

$$0\leqslant a_{ij}\leqslant m_j+p-j-\sum_{k=1}^{i-1}a_{kj} \qquad 0\leqslant b_{ij}\leqslant m_j-\sum_{k=1}^{r}a_{kj}-\sum_{k=1}^{i-1}b_{kj}$$

which arise from the fact that for a partition (standard or non-standard) $\alpha = (\alpha_1,\ldots,\alpha_p)$, $sp_\alpha$ is non-zero only if $\alpha_i \geqslant -(p-i)$.

### 5.2. Plethysms

Another application of the above VO construction is for calculating (outer) plethysms of symplectic and orthogonal $S$-functions. By the plethysm $f\otimes g$ of two symmetric functions $f$, $g\in\Lambda$, we mean express $g$ in terms of power sums and then make the substitution $p_j(x)\to f(x^j)$. The mapping $\psi_j : f(x)\to f(x^j)$ is normally called the Adam's operation. Much work has been done on the calculation of plethysms for $S$-functions (see [28] and references therein), and to a lesser extent for symplectic and orthogonal $S$-functions [33, 35]. Explicit formulae for the action of the Adam's operator $\psi_j$ on symplectic and orthogonal $S$-functions have been developed, in terms of the action of $\psi_j$ on normal $S$-functions. Here, we propose another method for calculating this action, by proceeding in the same vein as [32].

We now examine the details for the case of calculating $sp_\lambda(x^2)$. Begin by letting $\Gamma^{(2)}(z)$ denote the operator $\Gamma(z)$ given in (2.5) with $x\to x^2$, $z\to z^2$. The modes $\Gamma_i^{(2)}$ generate the function $sp_\lambda(x^2)$ in an analogous manner to (2.7). Note that we can write this as $\Gamma^{(2)}(z) = \hat{\Gamma}(z)\xi(z)$ where

$$\hat{\Gamma}(z) \equiv \Gamma(z)\Gamma(-z) = 2(1+z^2)\exp\left(2{\sum_j}' \frac{p_j}{j} z^j\right)\exp\left(-2{\sum_j}' \frac{\partial}{\partial p_j}(z^j + z^{-j})\right) \times \mathrm{e}^{\mathrm{i}2q}(-1)^{\alpha_0} z^{2\alpha_0+1}$$

$$\xi(z) = \frac{z}{2(1+z^2)}\,\mathrm{e}^{-\mathrm{i}q}(-1)^{\alpha_0-1}\exp\left({\sum_j}' \frac{\partial}{\partial p_j}(z^j + z^{-j})\right)$$

with ${\sum_j}' \equiv \sum_{j\,\mathrm{even}}$. Thus, if we express $sp_\lambda(x^2)$ as (the integral of) a product of currents $\Gamma^{(2)}(z_i)$ acting on **1**, we can move the operators $\xi(z_i)$ across the operators $\hat{\Gamma}(z_i)$ using the relation

$$\xi(z)\hat{\Gamma}(w) = \frac{-w^2}{(1-w^2z^2)(1-w^2/z^2)}\hat{\Gamma}(w)\xi(z).$$

These operators $\xi(z_i)$ will then act on **1** producing factors of $\mathrm{e}^{\mathrm{i}q}$, leaving just the currents $\hat{\Gamma}(z) = \Gamma(z)\Gamma(-z)$. For the simplest case of a one-part partition, we have

$$sp_{(n)}(x^2) = \int \frac{\mathrm{d}z}{z}\frac{-z^{1-2n}}{(1+z^2)}\hat{\Gamma}(z)\,\mathrm{e}^{-\mathrm{i}q} = \tfrac{1}{2}\sum_{j\geqslant 0}(-1)^{j+1}\hat{\Gamma}_{n-j}\,\mathrm{e}^{-\mathrm{i}q}.$$

However, if we expand $\hat{\Gamma}(z) = \sum_{n\in\mathbb{Z}}\hat{\Gamma}_n z^{2n-1}$, then $\hat{\Gamma}_n = \sum_{j\leqslant n-1}(-1)^j\Gamma_{2n-1-j}\Gamma_j$. Hence

$$sp_{(n)}(x^2) = \sum_{j\geqslant 0}\sum_{k=-1}^{n-j-1}(-1)^{j+k+1}\Gamma_{2n-2j-k-1}\Gamma_k\,\mathrm{e}^{-\mathrm{i}q} = \sum_{j=0}^{n}\sum_{k=0}^{n-j}(-1)^{j+k}sp_{(2n-2j-k,k)}. \tag{5.2}$$

From (5.2) and the result

$$sp_{(n)}^2 = \sum_{j=0}^{n}\sum_{k=0}^{n-j}sp_{(2n-2j-k,k)}$$

we have the plethysm

$$\begin{aligned} sp_{(n)} \otimes sp_{(2)} &\equiv \tfrac{1}{2}(sp_{(n)}(x^2) + sp_{(n)}^2) \\ &= sp_{(2n)} + sp_{(2n-2,2)} + \cdots + \frac{(1+(-1)^n)}{2}sp_{(n,n)} + sp_{(2n-3,1)} + sp_{(2n-5,3)} + \cdots \\ &\quad + \frac{(1+(-1)^n)}{2}sp_{(n-1,n-1)} + \cdots + \frac{(1+(-1)^{n-1})}{2}sp_{(2)} + \frac{(1+(-1)^n)}{2}sp_{(1,1)} \\ &\quad + \frac{(1+(-1)^n)}{2}sp_{(0)}. \end{aligned}$$

This procedure can be generalized in exactly the same way as was done in [32] to enable one to calculate $sp_\lambda(x^r)$ for general $\lambda$ and $r$.

### *5.3. Hirota polynomials*

As a final application, we shall construct a generating function for certain Hirota polynomials, which have symplectic $S$-functions as tau functions. We do this by mimicking the standard construction of the KP hierarchy as the orbit of $|0\rangle$ (or equivalently **1** in the modal language) under the group associated with $gl(\infty)$, the only difference being that we use the modified bras $|p\rangle'$ instead of the usual ones.

We begin by noting that equations (1.21) in [1] become

$$\langle p|'\psi(k)\,\mathrm{e}^{H(x)}a|0\rangle = k^{p-1}\langle p-1|'\mathrm{e}^{H(x-\epsilon(k^{-1})-\epsilon(k))}a|0\rangle$$
$$\langle p|'\psi^*(k)\,\mathrm{e}^{H(x)}a|0\rangle = (1-k^2)k^p\langle p1|'\mathrm{e}^{H(x+\epsilon(k^{-1})+\epsilon(k))}a|0\rangle. \tag{5.3}$$

To see this, just do the corresponding modal calculation using the result (4.6). The identities (5.3) permit one to derive the bilinear identity for the hierarchy in the usual way, which we now do. It is important to note that since we are using standard free fermions and the standard kets $|p\rangle$, the crucial first step remains unchanged in that the following equation is still valid:

$$\sum_{i\in\mathbb{Z}}\psi_i g|p\rangle\otimes\psi_i^* g|p\rangle = 0.$$

We then apply $\mathrm{e}^{H(x)}\otimes\mathrm{e}^{H(\bar{x})}$ and take the inner product with $\langle p+1|'\otimes\langle p-1|'$, getting (upon application of (5.3))

$$\int\frac{\mathrm{d}k}{k}(1-k^2)\,\mathrm{e}^{\xi(x-\bar{x},k)}\tau(x-\epsilon(k^{-1})-\epsilon(k))\tau(\bar{x}+\epsilon(k^{-1})+\epsilon(k)) = 0 \tag{5.4}$$

where $\epsilon(k) = (k, k^2/2, k^3/3, \ldots)$ and $\xi(x,k) = \sum_{j\geqslant 1}x_jk^j$. With the usual transformation $x\to x+y$, $\bar{x}\to x-y$ this can be rewritten in Hirota form, namely

$$\sum_{i,q\geqslant 0}S_{(i)}(-2y)S_{(q)}(\tilde{D})\left(S_{(i+q+1)}(\tilde{D})-S_{(i+q+3)}(\tilde{D})\right)\exp\left(\sum_n y_nD_n\right)\tau\cdot\tau = 0.$$

Here $S_\lambda(x)$ is the polynomial obtained from the $S$-function $s_\lambda(x)$ by replacing $p_j(x)\to jx_j$. Changing variables in the same way as was done in [36] one obtains the Hirota equations $P_\lambda(D)\tau\cdot\tau = 0$ with

$$P_\lambda(D) = \sum_{i,q\geqslant 0}S_{\lambda/(i)}(-\tilde{D}/2)S_{(q)}(\tilde{D})(S_{(i+q+1)}(\tilde{D})-S_{(i+q+3)}(\tilde{D})). \tag{5.5}$$

As one can see, although the sum over $i$ is finite (stopping at $i = |\lambda|$), $q$ is not constrained at all. Thus the Hirota polynomials are (inhomogeneous) of infinite order.

*Example.* Setting $\lambda = (1^3)$ in (5.8), you get

$$\begin{aligned}P_{(1^3)}(D) = {}& S_{(1^3)}(-\tilde{D}/2)((S_{(1)}(\tilde{D})-S_{(3)}(\tilde{D}))+S_{(1)}(\tilde{D})(S_{(2)}(\tilde{D})-S_{(4)}(\tilde{D}))+\cdots)\\ &+S_{(1^2)}(-\tilde{D}/2)((S_{(2)}(\tilde{D})-S_{(4)}(\tilde{D}))+S_{(1)}(\tilde{D})(S_{(3)}(\tilde{D})-S_{(5)}(\tilde{D}))+\cdots).\end{aligned} \tag{5.6}$$

Let us examine polynomial $\tau$ functions of order three. Due to the nature of the Hirota derivatives $D$, we only have to take terms up to order six in the above expression. Ignoring the odd degree terms in the resulting expression yields the Hirota equation

$$(24D_1^4+72D_2^2-96D_1D_3+5D_1^6+27D_1^2D_2^2-28D_1^3D_3+32D_3^2-36D_2D_4)\tau\cdot\tau = 0. \tag{5.7}$$

Note that the first three terms constitute a multiple of the normal degree four KP equation. If $\sigma = (\sigma_1,\ldots,\sigma_n)$, then let

$$\tau_\sigma = \frac{\partial^n\tau}{\partial\sigma_1\partial\sigma_2\cdots\partial\sigma_n}.$$

The above Hirota derivatives have the following action:

$$
\begin{aligned}
&D_1^4\tau\cdot\tau = 2\tau_{1^4}\tau - 8\tau_{1^3}\tau_1 + 6\tau_{1^2}\tau_{1^2}\\
&D_2^2\tau\cdot\tau = 2\tau_{22}\tau - 2\tau_2\tau_2\\
&D_1D_3\tau\cdot\tau = 2\tau_{31}\tau - 2\tau_3\tau_1\\
&D_1^6\tau\cdot\tau = 2\tau_{1^6}\tau - 12\tau_{1^5}\tau_1 + 30\tau_{1^4}\tau_{1^2} - 20\tau_{1^3}\tau_{1^3}\\
&D_1^2D_2^2\tau\cdot\tau = 2\tau_{2^21^2}\tau - 4\tau_{2^21}\tau_1 - 4\tau_{21^2}\tau_2 + 2\tau_{22}\tau_{1^2} + 4\tau_{21}\tau_{21}\\
&D_1^3D_3\tau\cdot\tau = 2\tau_{31^3}\tau - 6\tau_{31^2}\tau_1 + 6\tau_{31}\tau_{1^2} - 2\tau_3\tau_{1^3}\\
&D_3^2\tau\cdot\tau = 2\tau_{3^2}\tau - 2\tau_3\tau_3\\
&D_4D_2\tau\cdot\tau = 2\tau_{42}\tau - 2\tau_4\tau_2.
\end{aligned}
$$

Using the above, you can check that the polynomials

$$
\begin{aligned}
&SP_{(3)} = S_{(3)} = \tfrac{1}{6}x_1^3 + x_2x_1 + x_3\\
&SP_{(21)} = S_{(21)} - S_{(1)} = \tfrac{1}{3}x_1^3 - x_3 - x_1\\
&SP_{(1^3)} = S_{(1^3)} - S_{(1)} = \tfrac{1}{6}x_1^3 - x_2x_1 + x_3 - x_1
\end{aligned}
$$

do indeed satisfy (5.7). In addition, one can see that the Schur polynomials, $S_{(21)}$ and $S_{(1^3)}$ are *not* solutions of (5.7).

Similarly, if you seek order four polynomial solutions, then taking the next terms in (5.6) gives

$$
\begin{aligned}
(240D_1^4 &+ 720D_2^2 - 960D_3D_1 + 50D_1^6 + 270D_1^2D_2^2 - 280D_1^3D_3 + 320D_3^2 - 360D_2D_4\\
&+4D_1^8 + 60D_1^4D_2^2 + 90D_2^4 - 20D_1^5D_3 - 240D_1D_2^2D_3 + 160D_1^2D_3^2\\
&+90D_1^2D_2D_4 - 144D_1^3D_5)\tau\cdot\tau = 0.
\end{aligned}
$$

One can check for example, that $SP_{(2,2)} = S_{(2,2)} - S_{(1,1)} = \frac{1}{12}x_1^4 + x_2^2 - x_1x_3 - \frac{1}{2}x_1^2 + x_2$ is a solution of the above Hirota equation. In fact, we see that all symplectic polynomials are tau functions for all Hirota polynomials (5.5). This is because from the above fermionic formalism, all the symplectic Schur polynomials are generated through successive application of the operators $r_i$, defined by Jimbo and Miwa [1, equation (2.11)].

## 6. Conclusions

We have constructed a realization of symplectic and orthogonal Schur functions in terms of certain vertex operators, the modes of which obey free fermionic relations. As a result, we have been able to develop procedures for carrying out multiplication and plethysms of these types of symmetric functions in terms of functions labelled by non-standard partitions. In addition, we have constructed a set of infinite-order Hirota equations which admit symplectic Schur polynomials as tau functions. It is not clear, however, as to whether this is no more than a formal construct, of any intrinsic value or meaning for that matter.

Indeed, the relationship between the hierarchy generated by (5.5) and the normal $*$KP hierarchies is unclear to the author. As it was generated from the orbit of the vacuum $|0\rangle$ under the group associated with $gl(\infty)$ and not $sp(\infty)$, it does not seem to have anything to do with the CKP hierarchy. We can only assume that it is obtainable through some non-trivial transformation of the normal KP hierarchy.

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## Appendix

In this appendix, we provide what seems to be a new proof of the Giambelli determinantal form (1.7) for symplectic $S$-functions. In order to do this, we first need to prove the identity

$$\det\left(\frac{1}{(1-z_iw_j)(1-w_j/z_i)}\right)_{n\times n} = \frac{\prod_{1\leqslant a<b\leqslant n}(1-w_aw_b)(w_a-w_b)(1-z_az_b)(z_a^{-1}-z_b^{-1})}{\prod_{k,l=1}^{n}(1-z_kw_l)(1-w_l/z_k)}. \tag{A.1}$$

We proceed by induction. The result is certainly true for $n=1$. Assume the result is true for determinants of size $2,3,\ldots,n-1$. Denote the determinant on the left by $D_n(w_1,\ldots,w_n;z_1,\ldots,z_n)$. We consider both sides of (A.1) to be functions of $w_1$. The idea is to show that both sides of have the same singularity structure. Certainly, they both have poles at the same points, namely $w=z_j^{\pm 1}$, $j=1,\ldots,n$. Let us now show they have the same residues at these poles. Expanding the left-hand side (LHS) along the top row, we have

$$D_n(w_1,\ldots,w_n;z_1,\ldots,z_n) = \sum_{j=1}^{n}\frac{(-1)^{j-1}}{(1-z_jw_1)(1-w_1/z_j)}D_{n-1}(w_2,\ldots,w_n;z_1,\ldots,\hat{z}_j,\ldots,z_n)$$

where $\hat{z}_j$ means $z_j$ is omitted. From this we see that

$$\mathrm{res}(\mathrm{LHS},z_j) = \frac{(-1)^jz_j}{(1-z_j^2)}D_{n-1}(w_2,\ldots,w_n;z_1,\ldots,\hat{z}_j,\ldots,z_n).$$

Turning to the right-hand side, let RHS $= f(w_1)g$, where

$$f(w_1) = \frac{(w_1-w_2)\cdots(w_1-w_n)(1-w_1w_2)\cdots(1-w_1w_n)}{(1-z_1w_1)\cdots(1-z_nw_1)(1-w_1/z_1)\cdots(1-w_1/z_n)}$$

$$g = \frac{\prod_{2\leqslant a<b\leqslant n}(1-w_aw_b)(w_a-w_b)\prod_{1\leqslant p<q\leqslant n}(1-z_pz_q)(z_p^{-1}-z_q^{-1})}{\prod_{k=1}^{n}\prod_{l=2}^{n}(1-z_kw_l)(1-w_l/z_k)}.$$

Then

$$\begin{aligned}\mathrm{res}(f,z_j) &= -z_j\frac{(z_j-w_2)\cdots(z_j-w_n)}{(1-z_j/z_1)\cdots(1-z_j/z_{j-1})(1-z_j/z_{j+1})\cdots(1-z_j/z_n)}\\ &\quad\times\frac{(1-z_jw_2)\cdots(1-z_jw_n)}{(1-z_1z_j)\cdots(1-z_nz_j)}\\ &= \frac{(-1)^jz_j}{(1-z_j^2)}\frac{(1-w_2/z_j)\cdots(1-w_n/z_j)}{(z_1^{-1}-z_j^{-1})\cdots(z_{j-1}^{-1}-z_j^{-1})(z_j^{-1}-z_{j+1}^{-1})\cdots(z_n^{-1}-z_j^{-1})}\end{aligned}$$

$$\times \frac{(1-z_j w_2)\cdots(1-z_j w_n)}{(1-z_1 z_j)\cdots(1-z_{j-1}z_j)(1-z_{j+1}z_j)\cdots(1-z_n z_j)}.$$

Hence

$$\begin{aligned}\operatorname{res}(\mathrm{RHS}, z_j) &= \frac{(-1)^j z_j}{(1-z_j^2)} \frac{\prod_{2\leqslant a<b\leqslant n}(1-w_a w_b)(w_a-w_b)\prod_{\substack{1\leqslant p<q\leqslant n\\ p,q\neq j}}(1-z_p z_q)(z_p^{-1}-z_q^{-1})}{\prod_{\substack{k=1\\ k\neq j}}^{n}\prod_{l=2}^{n}(1-z_k w_l)(1-w_l/z_k)} \\ &= \frac{(-1)^j z_j}{(1-z_j^2)} D_{n-1}(w_2,\ldots,w_n;z_1,\ldots,\hat{z}_j,\ldots,z_n)\end{aligned}$$

by the inductive assumption. Thus the LHS and RHS of (A.1) have the same residue at the poles $w_1 = z_j$. A similar calculation shows that both sides also have the same residue at the poles $w_1 = z_j^{-1}$. Thus Liouville's theorem tell us that LHS − RHS is a constant function, and since $\mathrm{LHS}_{w_1=w_2} = 0 = \mathrm{RHS}_{w_1=w_2}$, this must be zero. Hence LHS = RHS and the induction is complete. Note that by multiplying the $i$th row of the determinant on the left-hand side of (A.1) by $1-z_i^2$, we have the equivalent identity

$$\det\left(\frac{1-z_i^2}{(1-z_i w_j)(1-w_j/z_i)}\right)_{n\times n} = F(z;w) \tag{A.2}$$

where

$$F(z;w) = \prod_{p=1}^{n}(1-z_p^2)\frac{\prod_{1\leqslant a<b\leqslant n}(1-w_a w_b)(w_a-w_b)(1-z_a z_b)(z_a^{-1}-z_b^{-1})}{\prod_{k,l=1}^{n}(1-z_k w_l)(1-w_l/z_k)}.$$

To prove the Giambelli formula, note that from (2.9) we have

$$\begin{aligned}sp_\lambda &= (-1)^{r(r-1)/2}\int \frac{\mathrm{d}z\,\mathrm{d}w}{zw} z_1^{-j_1}\cdots z_r^{-j_r}(-w_1)^{-i_1}\cdots(-w_r)^{-i_r}\Gamma^*(z_1)\cdots \\ &\qquad \times\Gamma^*(z_r)\Gamma(w_1)\cdots\Gamma(w_r)\mathbf{1} \\ &= \int \frac{\mathrm{d}z\,\mathrm{d}w}{zw} z_1^{-j_1}\cdots z_r^{-j_r}(-w_1)^{-i_1}\cdots(-w_r)^{-i_r}F(z;w) \\ &\qquad \times\exp\left(\sum_{j\geqslant 1}\frac{p_j}{j}(w_1^j+\cdots+w_r^j-z_1^j-\cdots-z_r^j)\right).\end{aligned}$$

Using (A.2) to convert $F(z;w)$ into a determinant, expressing the result as a sum over the symmetric group of the elements of the determinant, and then performing the various separate integrations using

$$sp_{(n-1|m)} = \int \frac{\mathrm{d}z\,\mathrm{d}w}{zw} z^{-n}(-w)^{-m}\frac{(1-z^2)}{(1-zw)(1-w/z)}\exp\left(\sum_{j\geqslant 1}\frac{p_j}{j}(w^j-z^j)\right)$$

and combining the results back into the determinant (see [2] for a similar calulation in the $S$-function case), we obtain the Giambelli formula (1.7).

## References


[1] Jimbo M and Miwa T 1983 *Publ. RIMS* **19** 943
[2] Jing N 1991 *J. Alg.* **138** 340
[3] You Y 1992 *J. Alg.* **145** 349
[4] Jing N 1991 *Adv. Math.* **87** 226
[5] Jing N H and Józefiak T 1992 *Duke Math. J.* **67** 377
[6] Jing N H 1994 *J. Alg. Comb.* **3** 291

[7] Bougourzi A H and Vinet L 1995 On the relation between $U_q(s\hat{l}(2))$ vertex operators and $q$-zonal functions CRM-2311
[8] Awata H, Odake S and Shiraishi J 1995 Integral representations of the macdonald symmetric functions q-alg 9506006
[9] Frenkel I B and Kac V G 1980 *Inv. Math.* **62** 23
[10] Kac V G, Kazhdan D A, Lepowsky J and Wilson R L 1981 *Adv. Math.* **42** 83
[11] Date E, Jimbo M, Kashiwara M and Miwa T 1982 *Publ. RIMS* **18** 1077
[12] ten Kroode F and van de Leur J 1991 *Commun. Math. Phys.* **137** 67
[13] Fenkel I B and Jing N 1988 *Proc. Natl Acad. Sci.* **85** 9373
[14] Frappat L, Sorba P and Scarrino A 1989 *J. Math. Phys.* **30** 2984
[15] Bougourzi A H and Weston R A 1994 *Int. J. Mod. Phys.* A **9** 4431
[16] Awata H, Odake S and Shiraishi J 1994 *Commun. Math. Phys.* **162** 61
[17] Skyrme T H R 1971 *J. Math. Phys.* **12** 1735
[18] Jarvis P D and Yung C M 1993 *J. Phys. A: Math. Gen.* **26** 5905
[19] Jarvis P D and Yung C M 1994 *J. Phys. A: Math. Gen.* **27** 903
[20] Hamel A M, Jarvis P D and Yung C M 1996 *J. Math. Comput. Mod.* to appear
[21] King R C 1990 *S*-functions and characters of Lie algebras and superalgebras *Invariant Theory and Tableaux* ed D Stanton (New York: Springer)
[22] King R C 1971 *J. Math. Phys.* **12** 1588
[23] Jarvis P D and Yung C M 1993 *J. Phys. A: Math. Gen.* **26** 1881
[24] Weyl H (ed) 1939 *The Classical Groups, Their Invariants and Representations* (Princeton, NJ: Princeton University Press)
[25] Koike K and Terada I 1987 *J. Alg.* **107** 466
[26] Samra N E and King R C 1979 *J. Phys. A: Math. Gen.* **12** 2305
[27] Hamel A M 1996 *Can. J. Math.* to appear
[28] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (Oxford: Oxford University Press)
[29] Ram A 1992 Weyl group symmetric functions and the representation theory of Lie algebras *Proc. 4th Conf. on Formal Power Series and Algebraic Combinatorics* pp 327–42
[30] Kac V G and Raina A K 1987 *Highest Weight Rxepresentations of Infinite-Dimensional Lie Algebras* (Singapore: World Scientific)
[31] Baker T H 1995 *J. Phys. A: Math. Gen.* **28** L331
[32] Baker T H 1995 *J. Phys. A: Math. Gen.* **28** 589
[33] Littlewood D E 1958 *Can. J. Math.* **10** 17
[34] Newell M J 1951 *Proc. R. Soc. Irish Acad.* **54** 153
[35] Scharf T and Thibon J-Y 1994 *Adv. Math.* **104** 30
[36] Baker T H, Jarvis P D and Yung C M 1993 *Lett. Math. Phys.* **29** 55